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ON THE CONTROL OF TRANSLATIONAL ROTATIONAL MOTION OF A SOLID BODY*

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The problem of control of the translational-rotational motion of a solid body induced by a system of n forces by controlling its apparent velocity is investigated. Algorithms which ensure asymptotic stability of such motion in the presence of constraints on the control parameters are presented. Generalized structures of algorithms for calculating the motion parameters of the solid body, which are necessary for determining the required control actions, are synthesized on the basis of available integral information about its motion.

1. Statement of the problem. We introduce two right-hand orthogonal bases, viz. base $\mathbf{E}$ rigidly attached to the solid body, whose axes do not generally coincide with the body principal central axes of inertia, and the inertial base $\mathbf{I}$. In the equations

$$\mathbf{r}^{\cdot} = \mathbf{V}, \quad \mathbf{V}^{\cdot} = \mathbf{A} + \mathbf{g}(\mathbf{r})$$

of motion of the solid body center of mass in the inertial space we shall consider the apparent velocity $\mathbf{W}$, i.e. the remainder of velocity $\mathbf{V}$ and its component $\mathbf{V_g}$ induced by the acceleration of gravity $\mathbf{g}(\mathbf{r})$ /1/. In the base $\mathbf{E}$ vector $\mathbf{W}$ conforms to the equation

$$\mathbf{W}^{\cdot} = \mathbf{a} - \boldsymbol{\omega} \times \mathbf{W} \tag{1.1}$$

where $\mathbf{a}$ is the representation of the apparent acceleration vector $\mathbf{A}$ in $\mathbf{E}$, and $\boldsymbol{\omega}$ is the vector of the body angular rotation velocity whose variation in time is defined by Euler's dynamic equations

$$J\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \mathbf{M} \tag{1.2}$$

On the assumption that motion of the body is controlled by a system of n forces of the form

$$\mathbf{F}_i = \mathbf{f}_i u_i, \quad |u_i| \leqslant 1$$

where $\mathbf{f}_i$ are vectors stationary in $\mathbf{E}$, u_i control parameters, and the subscript i assumes here and subsequently the values $1, 2, \ldots, n$; $n \geqslant 6$. Note that vectors $\mathbf{a}$ and $\mathbf{M}$ are defined in region U by the equations

$$\mathbf{a} = \sum_{i=1}^{n} \mathbf{a}_i u_i, \quad \mathbf{M} = \sum_{i=1}^{n} \mathbf{m}_i u_i; \quad U = \{\mathbf{u} : |u_i| \leqslant 1\} \tag{1.3}$$
$$(\mathbf{a}_i = m^{-1}\mathbf{f}_i, \quad \mathbf{m}_i = \boldsymbol{\rho}_i \times \mathbf{f}_i)$$

where m is the mass of the body, and $\boldsymbol{\rho}_i$ is the position vector of the point of application of force $\mathbf{F}_i$.

We specify in the inertial space $\mathbf{I}$ the apparent velocity vector $\mathbf{W}_*$ and in base $\mathbf{E}$ the fixed unit vector $\boldsymbol{\xi}$.

We have the problem of formulating control $\mathbf{u}$ under the constraint $\mathbf{u} \in U$ so as to impart to the rigid body the apparent velocity $\mathbf{W}$ which would coincide in the inertial space with $\mathbf{W}_*$ and be collinear with the unit vector $\boldsymbol{\xi}$ in base $\mathbf{E}$.

2. Synthesis of the control algorithm. The problem stated above reduces to that of controlling two processes, viz. the uniaxial orientation of the solid body along the fixed in $\mathbf{I}$ s-direction of the unit vector $\mathbf{s} = \mathbf{W}_* / \|\mathbf{W}_*\|$, and the process of imparting to the body the specified magnitude of the apparent velocity $\|\mathbf{W}_*\|$ along the direction. We construct $\mathbf{u}$ so as to simultaneously control both processes.

Note that the motion of unit vector $\mathbf{s}$ in base $\mathbf{E}$ satisfies the equation

$$\mathbf{s}^{\cdot} = -\boldsymbol{\omega} \times \mathbf{s} \tag{2.1}$$

We introduce vector $\mathbf{w} = \mathbf{W} - \mathbf{W}_*$. Taking into account that vector $\mathbf{W}_*$, which is stationary in $\mathbf{I}$, is time dependent in base $\mathbf{E}$ in conformity with the equation

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$$\mathbf{W}_*^{\cdot} = -\boldsymbol{\omega}\times\mathbf{W}_* \tag{2.2}$$

we obtain from (1.1) and (2.2) that $\mathbf{w}$ conforms in $\mathbf{E}$ to the equation

$$\mathbf{w}^{\cdot} = \mathbf{a} - \boldsymbol{\omega}\times\mathbf{w} \tag{2.3}$$

The problem formulated in Sect.1 is solved when the control $\mathbf{u}\in U$ brings the body to the state

$$\mathbf{w}=0,\quad \boldsymbol{\xi}=\mathbf{s},\quad \boldsymbol{\omega}=0 \tag{2.4}$$

is obtained.

We introduce the Liapunov function of the form

$$2V = \mu(\boldsymbol{\xi}-\mathbf{s})^2 + \boldsymbol{\omega}'J\boldsymbol{\omega} + \nu\mathbf{w}'\mathbf{w},\quad \mu>0,\quad \nu>0$$

which is positive for $\mathbf{w}\neq 0, \boldsymbol{\xi}\neq\mathbf{s}, \boldsymbol{\omega}\neq 0$ and vanishes in the equilibrium position of (2.4) of system (1.2), (2.1), (2.3). The time derivative of V is by virtue of Eqs. (1.2), (2.1) and (2.3) of the form

$$V^{\cdot} = \boldsymbol{\omega}'(\mathbf{M}-\mu\boldsymbol{\xi}\times\mathbf{s}) + \nu\mathbf{w}'\mathbf{a} \tag{2.5}$$

Assuming that vectors $\mathbf{M}$ and $\mathbf{a}$ in (2.5) are, respectively,

$$\begin{aligned}&\mathbf{M}\equiv\mathbf{M}_* = K\boldsymbol{\omega}+\mu\boldsymbol{\xi}\times\mathbf{s},\quad K<0,\quad \mathbf{M}_*=\{M_j^*\}\\&\mathbf{a}\equiv\mathbf{a}_*=Q\mathbf{w},\quad Q<0,\quad \mathbf{a}_*=\{a_j^*\}\quad (j=1,2,3)\end{aligned} \tag{2.6}$$

we have instead of (2.5)

$$V^{\cdot} = \boldsymbol{\omega}'K\boldsymbol{\omega} + \nu\mathbf{w}'Q\mathbf{w} \tag{2.7}$$

whose right-hand side is, as a function of vector $\mathbf{y}=\{\mathbf{w}', \mathbf{s}', \boldsymbol{\omega}'\}$, negative and of constant sign, since it vanishes not only in the equilibrium position (2.4) of the system considered here but, also, on the set

$$\begin{aligned}&Y = Y_1\cup Y_2;\quad Y_1=\{\mathbf{y}: \mathbf{w}=\boldsymbol{\omega}=0,\quad \mathbf{s}=-\boldsymbol{\xi}\},\\&Y_2=\{\mathbf{y}: \mathbf{w}=\boldsymbol{\omega}=0,\quad \mathbf{s}\neq\pm\boldsymbol{\xi}\}\end{aligned}$$

Since Y_1 represents the unstable equilibrium position of system (1.2), (2.1), (2.3) and the set Y_2 does not contain entire trajectories, control (2.6) guarantees according to the Barbashin–Krasovskii theorem the asymptotic stability position (2.4).

We have the problem of selecting the control $\mathbf{u}$ that would yield vectors $\mathbf{M}_*$ and $\mathbf{a}_*$ in conformity with algorithm (2.6).

Let the weight coefficients in the first of relations (2.6) be selected so that the condition

$$\mathbf{M}_*\in\mathbf{M}_\omega=\{\mathbf{M}: |u_i|\leqslant 1\} \tag{2.8}$$

is satisfied in the considered variation range of $\mathbf{s}$ and $\boldsymbol{\omega}$.

The problem of distribution of required control actions between parameters u_i can then be reduced to the problem of linear programming: determine parameters u_i which under linear constraints of the form

$$\sum_{i=1}^{n}\mathbf{m}_i u_i = \mathbf{M}_*,\quad \boldsymbol{\lambda}\times\sum_{i=1}^{n}\mathbf{a}_i u_i = 0,\quad \boldsymbol{\lambda}=\frac{\mathbf{a}_*}{\|\mathbf{a}_*\|} \tag{2.9}$$

would provide the maximum of the linear form $l=\mathbf{a}'\boldsymbol{\lambda}$

$$l_* = \max_{\mathbf{u}\in U} l \tag{2.10}$$

The obtained from the solution of problem (2.9), (2.10) values of l_* and the parameters of control u_i° are then used for calculating the unknown control parameters u_i /2/.

$$u_i=\begin{cases}u_i^\circ, & \varkappa>1\\ \varkappa u_i^\circ, & 0\leqslant\varkappa\leqslant 1\end{cases},\quad \varkappa=l_*^{-1}\|\mathbf{a}_*\|$$

Note that when $\varkappa>1$ formula (2.7) assumes the form $V^{\cdot}=\boldsymbol{\omega}'K\boldsymbol{\omega}+\gamma\mathbf{w}'Q\mathbf{w}$, $\gamma=\nu l_*/\|Q\mathbf{w}\|>0$ which with $K<0$ and $Q<0$ ensures that the system tends to the equilibrium position (2.4).

3. Calculation of the solid body motion parameters. To determine the required values of vectors $\mathbf{M}_*$ and $\mathbf{a}_*$ in conformity with algorithm (2.6) it is necessary to have information about the current state of system (1.2), (2.1), (2.3) in base $\mathbf{E}$. If the relative position of bases $\mathbf{E}$ and $\mathbf{I}$ is defined by matrix X of directional cosines, then the represent ation $\mathbf{s_E}$ in base $\mathbf{E}$ of vector $\mathbf{s}$ known in $\mathbf{I}$ is defined by the relation $s_E = Xs$.

Let information on the motion of body in base $\mathbf{E}$ of the form

$$\mathbf{p}_{q+1} = \int_{t_q}^{t_q+h} \boldsymbol{\omega}(\tau)\, d\tau \quad (h = \text{const},\ q = 0, 1, \ldots) \tag{3.1}$$

$$\mathbf{b}_{N+1} = \int_{t_N}^{t_N+H} \mathbf{a}(\tau)\, d\tau \quad (H = \text{const},\ N = 0, 1, \ldots)$$

be measurable. In conformity with this information for the problems of calculation of the solid body motion parameters are dedicated, for example, the papers /3 – 5/.

Using the apparatus of Lie algebra and Lie groups, we investigate the possibility of obtaining a general structure of algorithms for calculating parameters of solid body orientation and its apparent velocity.

The Poisson equation

$$X^{\cdot} = -\Omega(t)\, X, \qquad X(t_0) = X_0 \tag{3.2}$$

whose skew-symmetric matrix $\Omega(t)$ composed of projections of vector $\boldsymbol{\omega}$ on the axes of base $\mathbf{E}$ can be represented in the form

$$X^{\cdot} = \left(\sum_{\alpha=1}^{3} \omega_\alpha(t)\, A_\alpha\right) X \tag{3.3}$$

The constant matrices

$$A_1 = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}, \quad A_2 = \begin{Vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad A_3 = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

satisfy the equalities

$$[A_1, A_2] = -A_3, \ [A_1, A_3] = A_2, \ [A_2, A_3] = -A_1 \tag{3.4}$$

where ($[A_i, A_j] = A_iA_j - A_jA_i$ are Lie brackets), form the basis L of Lie algebra.

We seek a solution of Eq. (3.3) in the form of the product of two matrices

$$X(t) = S(t)\, X_0 \tag{3.5}$$

where for small $|t - t_0|$ matrix $S(t)$ is defined by the formula /6,7/

$$S(t) = \prod_{\alpha=1}^{3} \exp(g_\alpha(t)\, A_\alpha) \tag{3.6}$$

Since on the other hand $S(t)$ is an orthogonal matrix, it can be represented in the form /8/

$$S(t) = \exp \Phi(t) \tag{3.7}$$

whose skew-symmetric matrix $\Phi(t)$ we represent in the form

$$\Phi(t) = \sum_{\alpha=1}^{3} \varphi_\alpha(t)\, A_\alpha \tag{3.8}$$

For establishing the relation of elements φ_α of matrix $\Phi(t)$ and functions g_α we use the Campbell – Hausdorf formula (e.g., /9/) according to which $e^A e^B$ is represented in the form e^C, and in the case of $AB \neq BA$

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] + \ldots \tag{3.9}$$

since from (3.6) – (3.9) and equalities (3.4) we have

$$\varphi_1 = g_1 - \frac{1}{2}\sigma_1 g_2 g_3 - \frac{1}{12} g_1 (g_2^2 + g_3^2) - \ldots \quad (1\ 2\ 3) \tag{3.10}$$

where (1 2 3) indicates that expressions for φ_2, φ_3 are obtained from (3.10) by cyclic transposition of subscripts, and $\sigma_1 = -\sigma_2 = \sigma_3 = 1$.

If φ_α ($\alpha = 1, 2, 3$) are considered as coordinates of vector $\boldsymbol{\varphi}$, the relation between φ_α and matrix S of directional cosines is defined by the formula

$$S = E - \frac{\sin\varphi}{\varphi}\Phi + \frac{1-\cos\varphi}{\varphi^2}\Phi^2, \quad \varphi = (\varphi'\varphi)^{1/2} \tag{3.11}$$

Note that the equations which define functions $g_\alpha(t)$ $(\alpha = 1, 2, 3)$ are obtained by differentiating (3.5) with respect to time and the use of formula /6,7/

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{6}[A, [A, [A, B]]] + \ldots$$

which holds for $A, B \in L$, and in this case coincide, apart the notation, with traditional kinematic equations of solid body motion in terms of Krylov's angles /3/.

Let us assume that the angular velocity in the neighborhood of point $t = t_0$ can be represented by the Taylor series

$$\omega(t) = \omega_0 + \omega_0^{\cdot}\tau + {}^1/_2\omega_0^{\cdot\cdot}\tau^2 + \ldots, \quad \tau = t - t_0, \quad \omega_0^{(s)} = \omega^{(s)}(t_0)$$

with vector φ represented then by the series

$$\varphi(t) = \mathbf{p}(t) + \frac{1}{12}\Omega_0\omega_0^{\cdot}\tau^3 + \frac{1}{24}\Omega_0\omega_0^{\cdot\cdot}\tau^4 + \frac{1}{720}(9\Omega_0\omega_0^{\cdot\cdot\cdot} + 6\Omega_0^{\cdot}\omega_0^{\cdot\cdot} + \Omega_0^2\omega_0^{\cdot\cdot} - \Omega_0^3\omega_0^{\cdot} + 3\Omega_0^{\cdot 2}\omega_0)\tau^5 + \ldots$$

$$\Omega_0 = \Omega(t_0), \quad \mathbf{p}(t) = \int_{t_0}^{t}\omega(\zeta)\,d\zeta = \omega_0\tau + \frac{1}{2}\omega_0^{\cdot}\tau^2 + \frac{1}{6}\omega_0^{\cdot\cdot}\tau^3 + \ldots$$

Taking into account that the primary information (3.1) is received at discrete instants of time, we seek a solution of Eq.(3.2) of the form $X_{q+k} = S_{q+k}X_q$, where S_{q+k} are increments of the matrix of directional cosines induced by motion of the body during the step kh $(k = 1, 2, \ldots)$.

The introduction of notation

$$\mathbf{p}_{I,i} = \mathbf{p}_{q+I} - \mathbf{p}_{q+i}; \quad \mathbf{p}_{q+\nu} = \operatorname{sign}\nu \int_{t_q}^{t_q+\nu h}\omega(\tau)\,d\tau$$

enables us to define the general structure of algorithms of third order accuracy in the form

$$\varphi_{q+k} = \mathbf{p}_{q+k} + \xi\mathbf{p}_{I1,\,i1}\times\mathbf{p}_{J1,\,j1}, \quad \xi = k^3/(6\mu_{11}), \quad \mu_{11}\neq 0 \tag{3.12}$$

and specify fourth order of simplest algorithms by the formula

$$\varphi_{q+k} = \mathbf{p}_{q+k} + \sum_{\alpha=1}^{2}\xi_\alpha\mathbf{p}_{I\alpha,\,i\alpha}\times\mathbf{p}_{J\alpha,\,j\alpha} \tag{3.13}$$

$$\xi_1 = \frac{k^3}{12}\,\frac{2\mu_{22} - 3k\mu_{21}}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}, \quad \xi_2 = \frac{k^3}{12}\,\frac{3k\mu_{11} - 2\mu_{12}}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}$$

with errors over the step kh equal, respectively,

$$\delta\varphi_{q+k} = \frac{k^3}{72\mu_{11}}(2\mu_{12} - 3k\mu_{11})\Omega_q\omega_q^{\cdot\cdot}h^4 + O(h^5) \tag{3.14}$$

$$\delta\varphi_{q+k} = \frac{1}{120}\Big\{\Big[5(\mu_{13}\xi_1 + \mu_{23}\xi_2) - \frac{3}{2}k^5\Big]\Omega_q\omega_q^{\cdot\cdot\cdot} + [10(\mu_{14}\xi_1 + \mu_{24}\xi_2) - k^5]\Omega_q^{\cdot}\omega_q^{\cdot\cdot} - \frac{k^5}{6}(\Omega_q^2\omega_q^{\cdot\cdot} - \Omega_q^3\omega_q^{\cdot} + 3\Omega_q^{\cdot 2}\omega_q)\Big\}h^5 + O(h^6)$$

The following notation has been introduced in formulas (3.12)–(3.14):

$$\mu_{\alpha\beta} = |i_\alpha I_\alpha|(i_\alpha^\beta - I_\alpha^\beta) - |I_\alpha j_\alpha|(I_\alpha^\beta - j_\alpha^\beta) - |I_\alpha J_\alpha|(I_\alpha^\beta - J_\alpha^\beta) - |i_\alpha j_\alpha|(i_\alpha^\beta - j_\alpha^\beta)$$

$$\mu_{\alpha 4} = |i_\alpha I_\alpha|\, i_\alpha I_\alpha(i_\alpha - I_\alpha) + |I_\alpha j_\alpha|\, I_\alpha j_\alpha(I_\alpha - j_\alpha) - |I_\alpha J_\alpha|\, I_\alpha J_\alpha(I_\alpha - J_\alpha) - |i_\alpha j_\alpha|\, i_\alpha j_\alpha(i_\alpha - j_\alpha)$$

$$\Omega_q = \Omega(t_q), \quad \omega_q = \omega(t_q) \quad (\alpha = 1, 2;\ \beta = 1, 2, 3)$$

The analysis of (3.14) shows that when condition

$$2\mu_{12} - 3k\mu_{11} = 0$$

is satisfied, set (3.12) contains a subset of algorithms of the fourth order of accuracy.

For constructing algorithms for calculating the apparent velocity in base $\mathbf{E}$ we use the solution

$$\mathbf{W}(t) = S(t)\Big[\mathbf{W}(t_0) + \int_{t_0}^{t}S^{-1}(\tau)\,\mathbf{a}(\tau)\,d\tau\Big] \tag{3.15}$$

of Eq.(1.1) and formula (3.11).

The set of algorithms of third order of accuracy derived on the basis of initial information (3.1) is determined by formula

$$\mathbf{W}_{N+m} = S_{N+m}\left[\mathbf{W}_N + \mathbf{b}_{N+m} + \sum_{\sigma=1}^{3} \alpha_\sigma \mathbf{p}_{I\sigma,\, i\sigma} \times \mathbf{b}_{J\sigma,\, j\sigma} + \alpha_4 \mathbf{p}_{N+\rho 1} \times (\mathbf{p}_{N+\rho 2} \times \mathbf{b}_{N+\rho 3})\right] \tag{3.16}$$

obtained on the assumption that in point $t = t_N$ vectors $\boldsymbol{\omega}$ and $\mathbf{a}$ have second derivatives with respect to time.

In (3.16) S_{N+m} is the increment of matrix X of directional cosines over the step mh. The coefficients α_σ ($\sigma = 1, 2, 3$) are obtained from the linear equation

$$\begin{aligned}
&\Gamma\boldsymbol{\alpha} = \mathbf{c}, \quad \Gamma = \{\gamma_{\mu\sigma}\}, \quad \boldsymbol{\alpha} = \{\alpha_\sigma\} \quad (\mu, \sigma = 1, 2, 3)\\
&\gamma_{1\sigma} = (|I_\sigma| - |i_\sigma|)(|J_\sigma| - |j_\sigma|),\\
&\gamma_{2\sigma} = (|I_\sigma| I_\sigma - |i_\sigma| i_\sigma)(|J_\sigma| - |j_\sigma|)\\
&\gamma_{3\sigma} = (|J_\sigma| J_\sigma - |j_\sigma| j_\sigma)(|I_\sigma| - |i_\sigma|), \quad \mathbf{c} = \| {}^1/_2 k^2\, {}^2/_3 k^3\, {}^1/_3 k^3 \|'\\
&\alpha_4 = {}^1/_6 k^3 / |\rho_1 \rho_2 \rho_3|
\end{aligned}$$

The error $\delta\mathbf{W}_{N+m}$ over the step mh calculated by formula (3.16) is defined by the relation

$$\delta\mathbf{W}_{N+m} = \delta S_{N+m}(h^4)\, \mathbf{W}_N + \boldsymbol{\theta}_{N+m}(h^4)$$

where $\boldsymbol{\theta}_{N+m}(h^4)$ is the error of calculation of the integral in (3.16) over the time interval $t \in [t_N, t_N + mh]$, and $\delta S_{N+m}(h^4)$ is the error of calculation of the increment S_{N+m} of matrix X over the same time interval. Explicit expressions for vector $\boldsymbol{\theta}_{N+m}(h^4)$ and matrix $\delta S_{N+m}(h^4)$ are not adduced owing to their unwieldiness.

4. **Example**. Let us investigate the process of controlling the translational rotational motion of a solid body when $n = 6$, taking the set

$$\begin{aligned}
&\mathbf{a}_1 = \mathbf{a}_2 = \| p_1 0 0 \|', \quad \mathbf{a}_3 = \mathbf{a}_4 = \| 0 p_2 0 \|', \quad \mathbf{a}_5 = \mathbf{a}_6 = \| 0 0 p_3 \|'\\
&\mathbf{m}_1 = -\mathbf{m}_2 = \| 0 \mu_2 0 \|', \quad \mathbf{m}_3 = -\mathbf{m}_4 = \| 0 0 \mu_3 \|', \quad \mathbf{m}_5 = -\mathbf{m}_6 = \| \mu_1 0 0 \|'
\end{aligned}$$

as the vectors $\mathbf{a}_i$ and $\mathbf{m}_i$ ($i = 1, 2, \ldots, 6$).

We introduce the notation

$$m_j^* = M_j^*/\mu_j, \quad f_j^* = a_j^*/p_j \quad (j = 1, 2, 3)$$

If the parameters of control u_i obtained with the use of relations

$$u_{1,2} = {}^1/_2 (f_1^* \pm m_1^*), \quad u_{3,4} = {}^1/_2 (f_2^* \pm m_2^*), \quad u_{5,6} = {}^1/_2 (f_3^* \pm m_3^*)$$

do not belong to set U, we obtain their values that satisfy condition $\mathbf{u} \in U$ using the results of Sect.2.

Since the rank of the system of constraints (2.9) is five, the linear form $l = \mathbf{a}'\boldsymbol{\lambda}$ is a function of a single variable, and the determination of its maximum l^* does not present difficulties.

The set Λ of possible orientations of unit vector $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$ in base $\mathbf{E}$ is in this case of the form of a combination of six subsets

$$\begin{aligned}
&\Lambda = \bigcup_{i=1}^{6} \Lambda_i\\
&\Lambda_1 = \{\boldsymbol{\lambda} : m_2^* \lambda_1 \geqslant 0,\ s_1^- |\lambda_2| \leqslant \Phi_3,\ s_2^- |\lambda_3| \leqslant \Phi_1,\ \lambda_1 \neq 0\}\\
&\Lambda_2 = \{\boldsymbol{\lambda} : m_2^* \lambda_1 < 0,\ s_1^+ |\lambda_2| \leqslant \Phi_3,\ s_2^+ |\lambda_3| \leqslant \Phi_1\}\\
&\Lambda_3 = \{\boldsymbol{\lambda} : m_3^* \lambda_2 \geqslant 0,\ s_3^- |\lambda_1| \leqslant \Phi_2,\ s_4^- |\lambda_3| \leqslant \Phi_1,\ \lambda_2 \neq 0\}\\
&\Lambda_4 = \{\boldsymbol{\lambda} : m_3^* \lambda_2 < 0,\ s_3^+ |\lambda_1| \leqslant \Phi_2,\ s_4^+ |\lambda_3| \leqslant \Phi_1\}\\
&\Lambda_5 = \{\boldsymbol{\lambda} : m_1^* \lambda_3 \geqslant 0,\ s_5^- |\lambda_1| \leqslant \Phi_2,\ s_6^- |\lambda_2| \leqslant \Phi_3,\ \lambda_3 \neq 0\}\\
&\Lambda_6 = \{\boldsymbol{\lambda} : m_1^* \lambda_3 < 0,\ s_5^+ |\lambda_1| \leqslant \Phi_2,\ s_6^+ |\lambda_2| \leqslant \Phi_3\}\\
&\Phi_j = \max(2 - m_j^*,\ 2 + m_j^*), \quad j = 1, 2, 3\\
&s_1^\pm = \frac{p_3}{p_2}\ s_2^\pm = \frac{p_1}{p_2}\left(\frac{2}{|\lambda_1|} \pm \frac{m_2^*}{\lambda_1}\right), \quad s_3^\pm = \frac{p_3}{p_1}\ s_4^\pm = \frac{p_2}{p_1}\left(\frac{2}{|\lambda_2|} \pm \frac{m_3^*}{\lambda_2}\right)\\
&s_5^\pm = \frac{p_2}{p_1}\ s_6^\pm = \frac{p_3}{p_1}\left(\frac{2}{|\lambda_3|} \pm \frac{m_1^*}{\lambda_3}\right)
\end{aligned}$$

each of which has its proper solution

$$\begin{aligned}
&\Lambda_1 : u_1^\circ = \lambda_1^\circ, \quad u_2^\circ = \lambda_1^\circ - m_2^*, \quad \overset{\circ}{u}_{3,4} = {}^1/_2 (s_1^- \lambda_2 \pm m_3^*),\\
&\overset{\circ}{u}_{5,6} = {}^1/_2 (s_2^- \lambda_3 \pm m_1^*)\\
&\Lambda_2 : u_1^\circ = \lambda_1^\circ + m_2^*, \quad u_2^\circ = \lambda_1^\circ, \quad \overset{\circ}{u}_{3,4} = {}^1/_2 (s_1^+ \lambda_2 \pm m_3^*),
\end{aligned}$$

$$\overset{\circ}{u}_{5,6} = {}^1/_2\,(s_2+\lambda_3 \pm m_1^*)$$

$$\Lambda_3 : \overset{\bullet}{u}_{1,2} = {}^1/_2\,(s_3-\lambda_1 \pm m_2^*), \quad u_3^\circ = \lambda_2^\circ, \quad u_4^\circ = \lambda_2^\circ - m_3^*,$$

$$\overset{\circ}{u}_{5,6} = {}^1/_2\,(s_4-\lambda_3 \pm m_1^*)$$

$$\Lambda_4 : \overset{\bullet}{u}_{1,2} = {}^1/_2\,(s_3+\lambda_1 \pm m_2^*), \quad u_3^\circ = \lambda_2^\circ + m_3^*, \quad u_4^\circ = \lambda_2^\circ,$$

$$\overset{\circ}{u}_{5,6} = {}^1/_2\,(s_4+\lambda_3 \pm m_1^*)$$

$$\Lambda_5 : \overset{\circ}{u}_{1,2} = {}^1/_2\,(s_5-\lambda_1 \pm m_2^*), \quad \overset{\circ}{u}_{3,4} = {}^1/_2\,(s_6-\lambda_2 \pm m_3^*),$$

$$u_5^\circ = \lambda_3^\circ, \quad u_6^\circ = \lambda_3^\circ - m_1^*$$

$$\Lambda_6 : \overset{\circ}{u}_{1,2} = {}^1/_2\,(s_5+\lambda_1 \pm m_2^*), \quad \overset{\circ}{u}_{3,4} = {}^1/_2\,(s_6+\lambda_2 \pm m_3^*),$$

$$u_5^\circ = \lambda_3^\circ + m_1^*, \quad u_6^\circ = \lambda_3^\circ$$

$$(\lambda_j^\circ = \lambda_j / |\lambda_j|)$$

which are valid for $|m_j^*| \leqslant 2$ $(j = 1, 2, 3)$.

The maximum values l_i^* of the linear form l in the above subsets are, respectively,

$$l^*_{1,2} = p_1 s_1^{\mp}, \quad l^*_{3,4} = p_1 s_3^{\mp}, \quad l^*_{5,6} = p_1 s_5^{\mp}$$

In modelling the control of a solid body motion whose orientation in the intertial space was defined by the quaternion $T = \{\tau_0, \tau_1, \tau_2, \tau_3\}$, the process of imparting the apparent velocity $W_1 = 20$ m/s along its longitudinal axis ($\xi = \{1,0,0\}$) was investigated. The initial state of the body and the stationary in base **I** vector s were respectively set as follows:

$$\mathbf{W}(0) = 0, \ \omega(0) = 0, \ T(0) = \{0.001;\ 0.3;\ 0.6;\ 0.74169\}$$

$$s = \{0.96987;\ 0.17098;\ -0.17361\}$$

The pattern of apparent velocity variation defined by projections W_j $(j = 1, 2, 3)$ on the axes of base **E**, of angular velocities $\omega_1, \omega_2, \omega_3$ of body rotation, of the cosine of the angle between vectors ξ and $s(l)$, and of parameters u_i $(i = 1, 2, \ldots, 6)$ in the motion process of the body can be seen in Fig.1, where curves with even numbers are shown for clarity by dash lines.

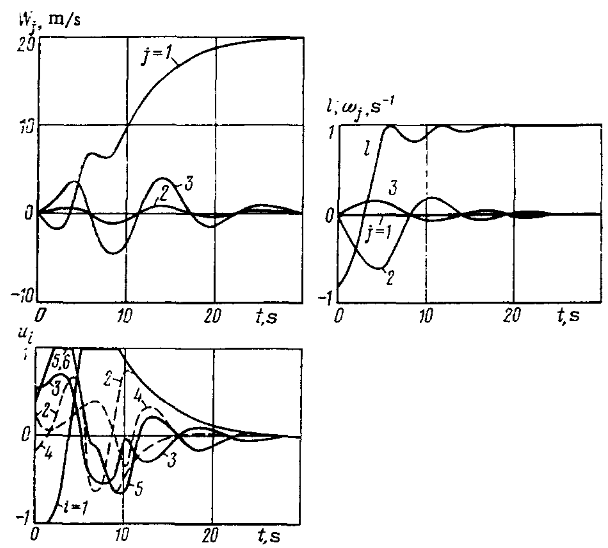

Fig.1

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